Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences
Tome 47, N° 4, 1994

MATHEMATIQUES
Théorie des probabilités

PROBABILISTIC ANALYSIS OF METHODS FOR NUMERICAL INTEGRATION*

P. Kopanov

(Submitted by Academician Bl. Sendov on December 29, 1993)

The general formulation of the problem under consideration is usually given as ollows.

Let $\mathbb{K} \subset \mathbb{R}^n$ be a compact set, $\mathbb{X} = \mathbb{X}(\mathbb{K})$ a functional space over $\mathbb{K}$ and $L : \mathbb{X} \to \mathbb{R}$, linear functional. A well known and an important problem is to compute the value $L(x)$ for any $x \in \mathbb{X}$. However, in general this problem is difficult since only for some pecial L and $\mathbb{X}$ one can find a convenient expression for $L(x)$. In some cases we an construct a sequence $\{L_n\}$ of linear approximators of L obeying the following two roperties:

i) For any $n \in \mathbf{N}$ and $x \in \mathbb{X}$ the computation of $L_n(x)$ is relatively easy (e.g. if $L_n(x)$ is given by a quadrature formula).

ii) $L_n(x) \to L(x)$ as $n \to \infty$ for any $x \in \mathbb{X}$.

Unfortunately, sometimes property ii) does not hold for all $x \in \mathbb{X}$ and for that roperty additional conditions for $\mathbb{X}$ are needed. Let us mention that in this area here is a problem called the stopping problem: Let $\varepsilon > 0$ and the integer k be fixed. f n is such that $|L_{n+k}(x) - L_n(x)| < \varepsilon$, then we stop the computation procedure nd take either $L_n(x)$ or $L_{n+k}(x)$ as an ε-approximation of $L(x)$. It is impossible, owever, to choose a stopping rule n appropriate for all $x \in \mathbb{X}$. In particular, if $M > 0$ and $n, k \in \mathbf{N}$ are fixed, it is possible to find $x \in \mathbb{X}$ such that $L(x) = M$ but $L_n(x) = \cdots = L_{n+k}(x) = 0$. Obviously, the solution of the stopping problem depends n the numerical method, i.e. on the sequence of approximators $\{L_n\}$, and on the roperties of the elements $x \in \mathbb{X}$.

Let us turn now to the construction of a function which is "bad" for any method of umerical integration over $C[0,1]$.

Let $\mathbb{X} = C[0,1]$, $L(x) = \int_0^1 x(t)\,dt$, $x \in C[0,1]$ and for $m = n, n+1, \ldots, n+k$, $\{L_m\}$ e a sequence of approximators for L, where $L_m(x) = \sum_{i=0}^{m} c_i^m x(t_i)$, $0 = t_0^m < t_1^m < t_2^m <$ $\cdots < t_m^m = 1$, $c_i^m \geq 0$, $\sum_{i=1}^{m} c_i^m = 1$. Below $M > 0$ is a fixed number. We construct $x(t)$, $t \in [0,1]$ as follows:

*This research work was partially supported by the Bulgarian Ministry of Science and Education under grant No MM–38/91.

a) $x(t) = 0$ for $t = t_j^m$, $j = 0, \ldots, m, m = n, n+1, \ldots, n+k$.

b) $x(t) = qM$ for $t = (t_i^n + t_j^m)/2$ if there are no other points of interpolation in the subinterval $[t_i^n, t_j^m]$.

c) $x(t)$ is linear between these points.

Obviously, $x \in C[0,1]$ and $L_m(x) = 0$ for $m = n, n+1, \ldots, n+k$. However, $L(x) = q.M/2$ and if we choose $q > 2$ then $L(x) > M$ and $L(x) - L_m(x) > M$, too.

Clearly, for the function x satisfying a), b), c), the stopping problem does not have a satisfactory solution. The same holds for polynomials approximating x (e.g. for Bernstein polynomials). Hence, more restrictions on $\mathbb{X}$ would eventually guarantee the smallness of the error in the stopping problem.

In any case, involving functions like the one described above, we can ask the following natural question: In the space $\mathbb{X}$, what is the portion of these "bad" functions, i.e. functions $x \in \mathbb{X}$, such that the computation of $L(x)$ can be performed with an error greater than fixed ε, $\varepsilon > 0$? A correct answer can be given in the framework of a well defined probability model and the problem is reduced to finding the probability of the event $\{|L(x) - L_n(x)| > \varepsilon\}$ for fixed $\varepsilon > 0$ and n.

If we know the probability $P\{|L(x) - L_n(x)| < \varepsilon\}$, we can compare different numerical methods, e.g. different sequences of approximators $\{L_n\}$. Of a great importance is any case when the above probability is "close" to 1 for any $\varepsilon > 0$. One of the aims of the present study is to find the probability $P\{|L(x) - L_n(x)| > \varepsilon\}$ for a given functional space $\mathbb{X}$, linear functional L and approximating sequence $\{L_n\}$. To reach that aim we have to pass the following three steps:

1) Specify the probability model $(\mathbb{X}, F, P)$, i.e. we have to define the probability measure P on the σ-algebra F (usually F is the Borel σ-algebra generated by the open subsets of $\mathbb{X}$).

2) Define the linear operator L and the sequence of approximators $\{L_n\}$ as random variables on the probability space $(\mathbb{X}, F, P)$.

3) Find, when possible, the distributions of the random variables L, L_n and of the error $D_n = L - L_n$.

If we know the distribution of D_n, we can find the probability $P\{|L - L_n| > \varepsilon\}$ exactly, or find its upper bound. Moreover, we can choose a probabilistic criterion for comparing different methods and then looking for the best method with respect to this criterion. One possibility is to estimate the variance $\mathrm{Var}(D_n)$ of the random error D_n. This criterion, being a natural extension of the Gauss least squares method, is widely used in stochastics and its applications.

Let us focus our attention to the space $C = C[0,1]$ of all continuous, real valued functions $x(t)$, $t \in [0,1]$ such that $x(0) = 0$. Let F denote the σ-algebra of Borel subsets of C, and P be the standard Wiener measure of F. The construction of the Wiener measure is well known and can be found in the literature, see e.g. HIDA [1] or LAMPERTI [4]. The natural process, defined on the probability space (C, F, P), denoted here by X_t, $t \geq 0$, is usually called a standard Wiener process.

It is well known that if $L : C[0,1] \to \mathbb{R}$ is a linear continuous functional then $L(x) = \int_0^1 x(t)\, d\varphi(t)$ where $\varphi \in V[0,1]$ is a function of bounded variation (This is exactly the Riesz theorem, e.g. see KOLMOGOROV and FOMIN [2]). Let now $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$, be a partition of $[0,1]$ with a diameter $d_n = \max\limits_{1 \leq i \leq n} (t_i - t_{i-1}) \to 0$ as $n \to \infty$ and let

$$T_n(x) = \sum_{i=1}^{n} x(t_i)\big[(\varphi(t_i) - \varphi(t_{i-1}))\big].$$

Then $T_n(x) \to L(x)$ for any $x \in C[0,1]$ and therefore the distribution of L is a limit of the distributions of T_n as $n \to \infty$. Thus the first step is to find the distribution of T_n. An easy transformation shows that

$$T_n = \sum_{i=1}^{n} X_{t_i}\left[(\varphi(t_i) - \varphi(t_{i-1}))\right] = \sum_{i=1}^{n} [\varphi(1) - \varphi(t_{i-1})].(X_{t_i} - X_{t_{i-1}}).$$

The last quantity is a sum of independent and normally distributed random variables where $X_{t_i} - X_{t_{i-1}} \sim N(0, t_i - t_{i-1})$. Hence

$$T_n \sim N(0, \sigma_n^2), \quad \text{where} \quad \sigma_n^2 = \sum_{i=1}^{n} \big(\varphi(1) - \varphi(t_{i-1})\big)^2.(t_i - t_{i-1}).$$

Thus $L \sim N(0, \sigma^2)$ with $\sigma^2 = \lim \sigma_n^2 = \int_0^1 (\varphi(1) - \varphi(t))^2\, dt$, $n \to \infty$.
Hence, we have established the following result.

Proposition 1. Let $L(x) = \int_0^1 x(t)\, d\varphi(t)$. Then the random variable L is normally distributed:

$$L \sim N(0, \sigma^2), \quad \sigma^2 = \int_0^1 \big(\varphi(1) - \varphi(t)\big)^2\, dt.$$

If $\varphi(t) = t$, then $L(x) = \int_0^1 x(t)\, dt$, and $L \sim N(0, 1/3)$. Further on we consider in more details just this case. Let now $\{L_n\}$ be a sequence of approximators of L and $L_n(x) = \sum_{i=1}^{n} c_i.x(t_i)$ with t_j are as above and $c_i \geq 0$, $c_0 + c_1 + \cdots + c_n = 1$. By the Riesz theorem $L_n(x) = \int_0^1 x(t)\, d\varphi_n(t)$ where $\varphi_n(t)$ is a step–wise function with jumps at the point t_i and size of the jumps proportional to c_i. The error $D_n(x) = L(x) - L_n(x)$ is also a linear functional and $D_n(x) = \int_0^1 x(t)\, d(t - \varphi_n(t))$. For D_n we obtain $D_n \sim N(0, \delta_n^2)$ where $\delta_n^2 = \int_0^1 (1 - \varphi_n(1) - t + \varphi_n(t))^2\, dt$. In other words we have found the distribution of the error D_n. Now we can compare the errors of using different quadrature formulas and look for the formula with the smallest variance of error. Since $\mathrm{Var}(D_n) = \delta_n^2 = \int_0^1 (1 - \varphi_n(1) - t + \varphi_n(t))^2\, dt$, we want to minimize this quantity. The following transformations hold:

$$\delta_n^2 = \int_0^1 \big(1 - t - (\varphi_n(1) - \varphi_n(t))\big)^2\, dt$$

$$= \int_0^1 (1-t)^2\, dt - 2\int_0^1 (1-t)(\varphi_n(1) - \varphi_n(t))\, dt + \int_0^1 \big(\varphi_n(1) - \varphi_n(t)\big)^2\, dt$$

$$= \frac{1}{3} - 2\sum_{i=1}^{n} d_i.\int_{t_{i-1}}^{t_i} (1-t)\, dt + \sum_{i=1}^{n} d_i^2.\int_0^1 dt$$

$$= \frac{1}{3} + \sum_{i=1}^{n} \left[d_i^2(t_i - t_{i-1}) + d_i.(t_i - t_{i-1}).(t_i + t_{i-1} - 2)\right],$$

where $d_i = \varphi_n(1) - \varphi_n(t)$ in subinterval (t_{i-1}, t_i).
Thus we obtain

$$\delta_n^2 \geq \frac{1}{3} - \frac{1}{4} \cdot \sum_{i=1}^{n} (t_i - t_{i-1}) \cdot (2 - t_i - t_{i-1})^2,$$

where an equality holds only if $2d_i = 1 - t_i - t_{i-1}$. This observation and the restriction $0 = t_0 < t_1 < \cdots < t_n = 1$ imply that:

$$t_i = \frac{i}{n}, \quad i = 0, 1, 2, \ldots, n, \quad c_0 = c_n = \frac{1}{2n}, \quad c_i = \frac{1}{n}, \quad i = 1, 2, \ldots, n-1, \quad \delta_n^2 = \frac{1}{(12n^2)}.$$

The quadrature formula based on these t_i, c_i corresponds to the well known trapezoidal method. Thus we have proved the following result.

Proposition 2. Among all the methods, the trapezoidal method has the smallest variance of error over the space $C[0, 1]$ with Wiener measure.

Remark. The above reasoning can be extended so to cover a case like $L(x) = \int_0^1 x(t)\, d\varphi(t)$ where x is a more general Gaussian process.

Acknowledgement. The author's thanks are addressed to Dr J. Stoyanov for proposing these topics and for his permanent encouragement and help.

REFERENCES

[1] HIDA T. Brownian motion. Springer–Verlag, Berlin, 1980. [2] KOLMOGOROV A.N., S.V. FOMIN. Introductory Real Analysis. Prentice–Hall, Englewood Cliffs (NY), 1970. [3] KUO H.-H. Gaussian Measures in Banach Spaces. Springer–Verlag, Berlin, 1975. [4] LAMPERTI J. Stochastic Processes. A Survey of the Mathematical Theory. Springer–Verlag, Berlin, 1977. [5] STORY D. P. Martingale Analysis, Technical Report, Univ. Ohio, Akron, 1984. [6] CAMBANIS S., E. MASRY. Sampling designs for the detection of signals in noise. IEEE Trans. Inform. Theory, **IT–29**, 1984, 83. [7] Id. Trapezoidal Monte Carlo Integration. SIAM J. Numer. Anal., **27**, 1990, 225.

Faculty of Mathematics
University of Plovdiv
4000 Plovdiv, Bulgaria